# Statistical and Computational Guarantees for Influence Diagnostics

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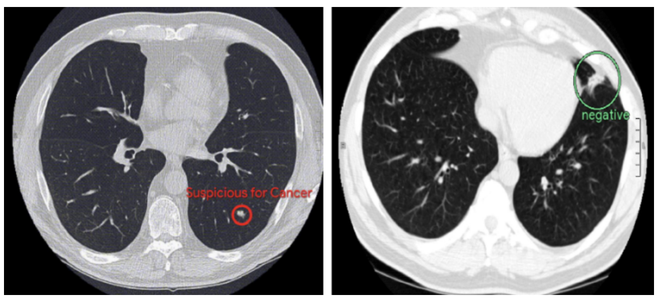
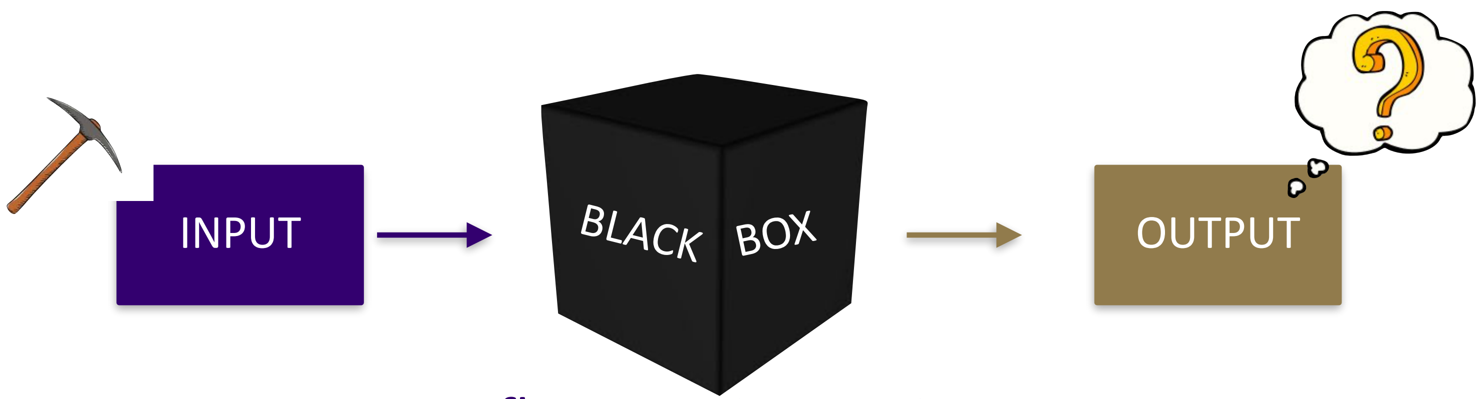
# *Motivation*

We rely on models for important tasks…




But how do we know we can <u>trust these models</u>?




Influence Diagnostics

# *Outline*

- **Background: Influential Points**
- Statistical Finite Bound
- Computational Bound
  - *Experiment: Is there always meaning?*
- Most Influential Subset
  - *Experiment: Are all statistics a lie?!*
- NLP Connection
  - *Will there be influence in your future?*

# *Background: Notation*

**Setting: Consider $\theta \in \Theta$, constructed from i.i.d sample $z = \{(x_i, y_i)\}_{i=1}^{n}$**

**True Parameter**

$$\theta_\star := \arg\min_{\theta\in\Theta} \mathbb{E}_{Z\sim P}\left[\ell(Z,\theta)\right]$$

**Estimator**

$$\theta_n := \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \ell\left(Z_i,\ \theta\right)$$

## Perturbed Estimator:

$$\theta_{n,\,\epsilon,\,z} := \arg\min_{\theta\in\Theta} \left\{ (1-\epsilon)\frac{1}{n}\sum_{i=1}^{n} \ell\left(Z_i,\ \theta\right) + \epsilon\ell(z,\theta) \right\}$$

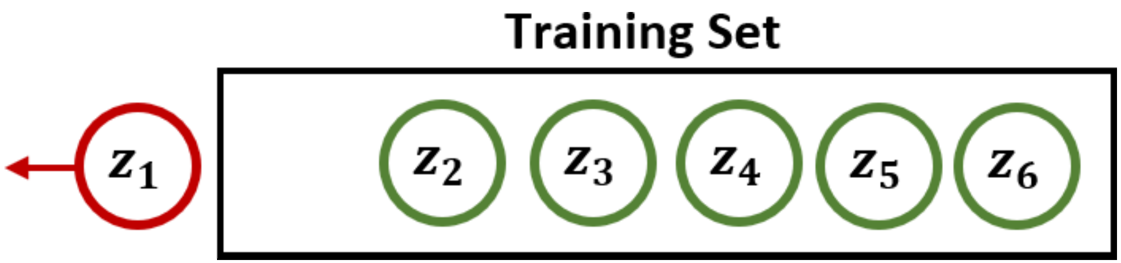
Empirical Risk

Loss at 1 point

$\varepsilon = \dfrac{-1}{n}$ → removing one point

# Background: Influence Function

Consider a prediction problem,




$$z_i = (x_i,\ y_i) \in X \times Y$$

**Empirical Risk Estimator**

$$\hat{\theta} \in \arg\min_{\theta \in \Theta} \frac{1}{6} \sum_{i=1}^{6} L(z_i, \theta)$$

**Perturbed Estimator**

$$\hat{\theta}_{z_1} \in \arg\min_{\theta \in \Theta} \frac{1}{6} \sum_{i=1}^{6} L(z_i, \theta) - \frac{1}{6} L(z_1, \theta)$$

**Parameter of Interest**

???

# *Background: Notation*

**Influence Function: quantify the *influence of a fixed data point* $z$ on an estimator $\theta_n$**

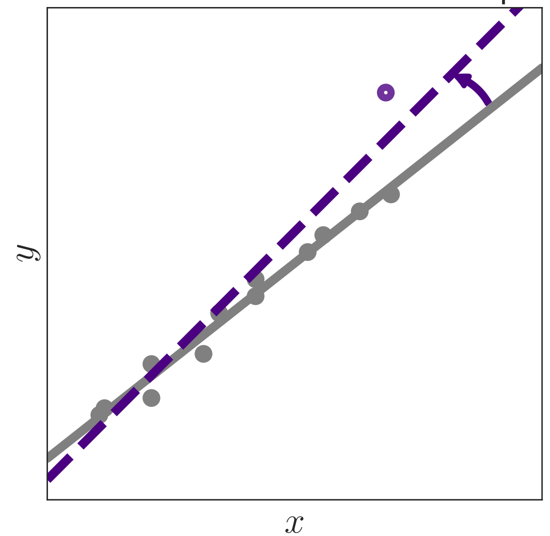
$$I_n(z) = \frac{d\theta_{n,\epsilon,z}}{d\epsilon} \approx \frac{\theta_{n,\epsilon,z} - \theta_n}{\epsilon}$$

**Cook and Weisberg Formula**

$$I_n(z) = -H_n(\theta_n)^{-1} \nabla \ell(z, \theta_n)$$

where $H_n(\theta_n)$ is the empirical Hessian

Illustration of Influence of a Datapoint




Low influence data point

Closed form solution —> easier to solve!

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# *Assumptions: Pseudo Self-Concordance*

1. Simple definition if we assume *linear prediction models (i.e.* $\ell(\theta) = \ell(Y, X^T\theta)$*).*

*We consider* $\ell(\theta)$ *is pseudo self-concordant if*

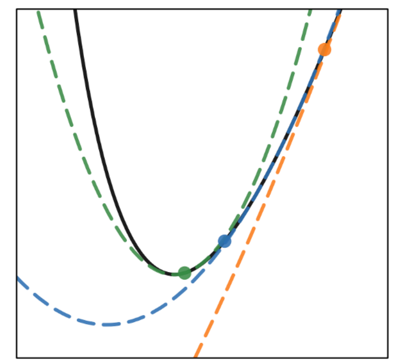
$$|\nabla^3 \ell(z, \theta)| \leq \nabla^2 \ell(z, \theta)$$

**Prevents** $\nabla^2 \ell(z, \theta)$ **from changing too quickly with** $\theta$

**Consequence: Spectral Approximation of the Hessian**

$$\frac{1}{2} H(\theta') \leq H(\theta) \leq 2H(\theta')$$ **for** $\theta$ **close to** $\theta'$

Illustration of Pseudo Self-Concordance



Black curve: population function $f(x)$; colored dot: reference point $x_i$; colored dashed curve: quadratic approximation at the corresponding reference point $Q(x; x_i)$.

# *Assumptions*

2. **Normalized gradient** $H(\theta_\star)^{-1/2}\nabla \ell(Z, \theta_\star)$ **at** $\theta_\star$ **is sub-Gaussian** with parameter $K_1$

**Since** $\mathbb{E}[\nabla \ell(Z, \theta_\star)] = 0$, **then Assumption 2 gives a high prob. bound on** $\|\nabla \ell(Z, \theta_\star)\|_{H_\star}^{-1}$

3. There exist $K_2 > 0$ such that the **standardized Hessian at** $\theta_\star$ **satisfies a Bernstein condition** with parameter $K_2$

Moreover,

$$\sigma_H^2 := \|\mathrm{Var}(H(\theta_\star)^{-1/2}\nabla^2 \ell(Z, \theta_\star) H(\theta_\star)^{-1/2})\|_2 \text{ is finite.}$$

**Assumption 3 gives spectral concentration**

$$(1/2)H(\theta) \prec H_n(\theta) \prec 2H(\theta)$$

**Generalized Linear Models satisfy these assumptions**

# *Results: Statistical Bound*

**Theorem 1. Suppose the assumptions[1] hold and**

$$n \geq C\left(\frac{p}{\mu_\star}\log\frac{1}{\delta} + \log\frac{p}{\delta}\right)$$

**where** $\mu_\star = \lambda_{\min}(H(\theta_\star))$**.**

**Then, with probability at least** $1-\delta,$ **we have** $\frac{1}{4}H(\theta_\star) \leq H_n(\theta_n) \leq 3H(\theta_\star)$ **and**

$$\|I_n(z) - I(z)\|_{H_\star}^2 \leq C\frac{p_\star^2}{\mu_\star n}\text{poly}\log\left(\frac{p}{\delta}\right)$$

- Only logarithmic dependence on $p$ (dim. of param.)
- $p_\star$ is the degrees of freedom (model misspecification)
- Rate of $1/n$

1. Assumptions met by Generalized Linear Models

# *Experiment: Simulation*

**Simulation**

$x \sim N(0,1)$

Linear (Ridge) Regression

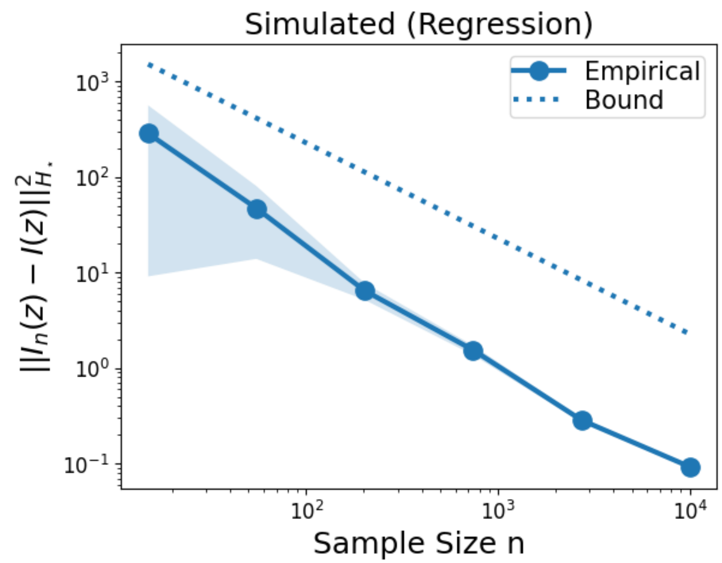
Logistic Regression

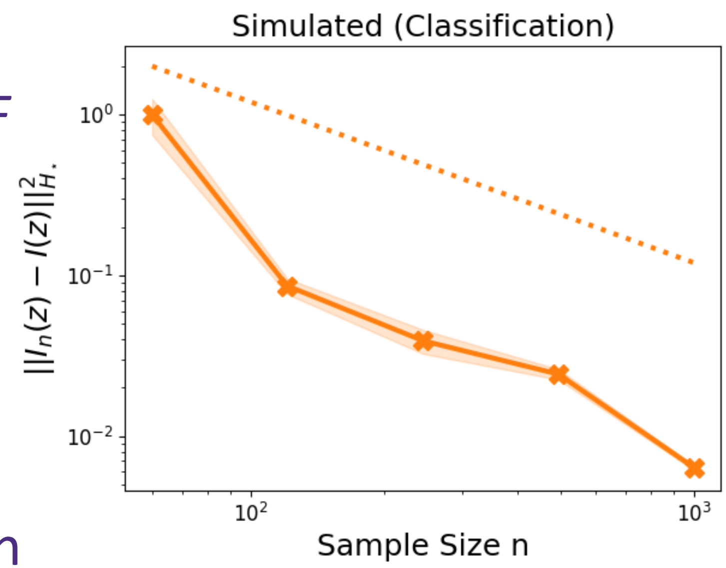
*X-axis: Training Sample Size*

*Y-axis: Difference in empirical vs. population IF*

**Results**

- See 1/n of our bound observed
- Straight line in log-log scale
- Hard to approximate classification population



# *Experiment: Real Dataset*

**Real Dataset**

Cash Transfer

- X: Socio-economic covariates
- Y: Total consumption (regression)

Oregon Medicaid

- X: Health-related covariates

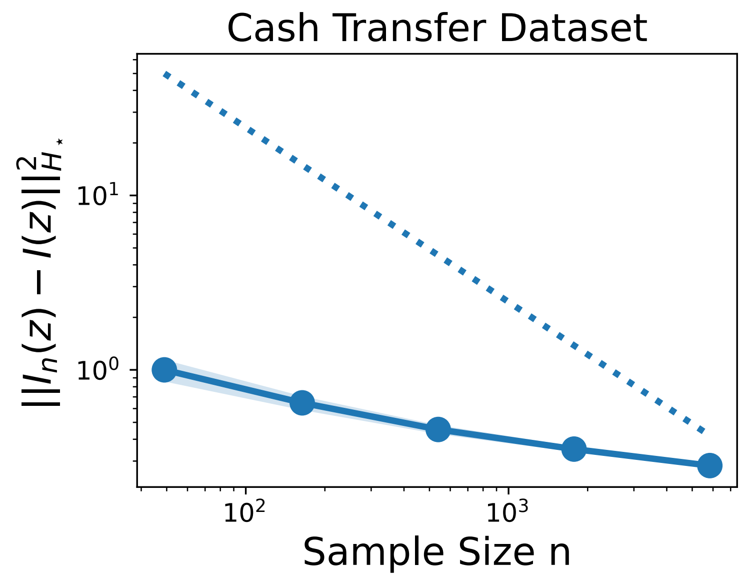
1. Y: Estimate overall health (classification)
2. Y: Number of good days (regression)

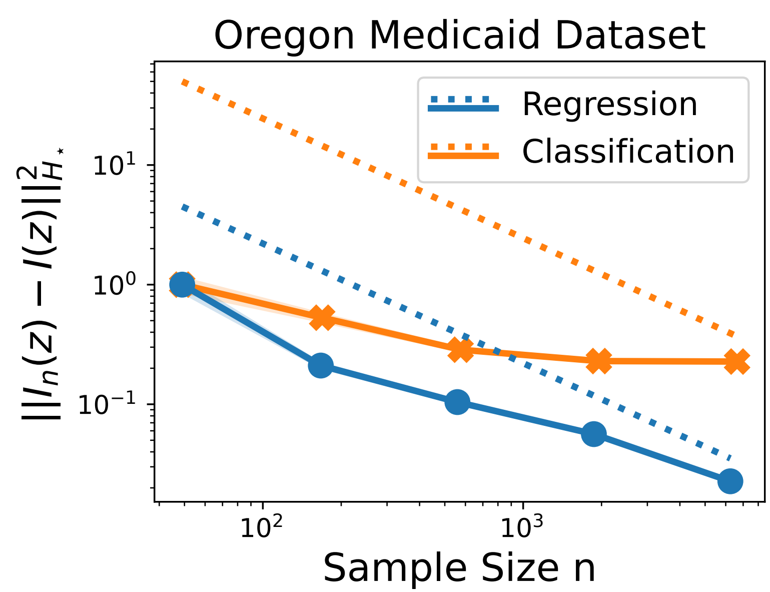
*X-axis: Training Sample Size*

*Y-axis: Difference in empirical vs. population IF*

**Results**

- See 1/n of our bound observed
- Straight line in log-log scale
- Hard to approximate classification population



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# Computational Challenge

Second derivative (p x p)
p = dim of parameter

**Cook and Weisberg Formula**

$$I_n(z) = -H_n(\theta_n)^{-1} \nabla \ell(z, \theta_n)$$

Can't be computed for large values of $p$

**Instead use iterative algorithms to approximately minimize**

$$g_n(\mu) := \frac{1}{2}\left\langle \mu, H_n(\theta_n)\mu \right\rangle + \left\langle \nabla \ell(z, \theta_n), \mu \right\rangle$$

**Algorithms**

> Conjugate Gradient (CG)

> Stochastic Gradient Descent (SGD)

> Stochastic Variance Reduced Gradient (SVRG)

> Arnoldi – Low Rank

# Result: Computational Bound

**Proposition 1. Consider the setting of Theorem 1, and let $\mathscr{G}$ denote the event under which its conclusions hold. Let $\hat{I}_n(\theta)$ be an estimate of $I_n(\theta)$ that satisfies**

$$\mathbb{E}_{Z_{1:n}}\left[\left\|\hat{I}_n(z) - I_n(z)\right\|^2_{H_n(\theta_n)}\right] \leq \epsilon.$$

**Then**

$$\mathbb{E}_{\mathscr{G}}\left[\left\|\hat{I}_n(z) - I(z)\right\|^2_{H(\theta_\star)}\right] \leq 8\epsilon + C\frac{p_\star^2}{\mu_\star n}\text{poly}\log\frac{p}{\delta}$$

- Using an $\epsilon$-approximate minimizer of the empirical influence approximation
- Translating approx. error in $H_n(\theta_n)$-norm to the $H_\star$-norm under $\mathscr{G}$ (Theorem 1)
- **Total Error** under $O(\epsilon)$ is $O(n(\epsilon)T(\epsilon))$

Proposition 1

Statistical (Thm. 1)

Computational

# *Result: Computational Bound*

**Proposition 1. Consider the setting of Theorem 1, and let $\mathscr{G}$ denote the event under which its conclusions hold. Let $\hat{I}_n(\theta)$be an estimate of $I_n(\theta)$ that satisfies**

$$\mathbb{E}_{Z_{1:n}}\left[\left\|\hat{I}_n(z) - I_n(z)\right\|^2_{H_n(\theta_n)}\right] \leq \epsilon.$$

**Then**

$$\mathbb{E}_{\mathscr{G}}\left[\left\|\hat{I}_n(z) - I_n(z)\right\|^2_{H(\theta_\star)}\right] \leq 8\epsilon + C\frac{p_\star^2}{\mu_\star n}\text{poly}\log\frac{p}{\delta}$$

<u>**Example: Stochastic Variance Reduction Gradient (SVRG)**</u>

- Requires $T_n(\epsilon) = C(n+\kappa_n)\log\left(\frac{\kappa_n\|u_0 - u_\star\|_{H_n(\theta_n)}}{\epsilon}\right)$iterations to return an $\epsilon$-approximate minimizer.
- Each iteration requires $n$ Hessian-vector products
- To make statistical error to be smaller than $\epsilon$, $n \geq n(\epsilon) = \tilde{O}\left(\frac{p_\star^2}{\mu_\star\epsilon}\right)$ from **Theorem 1**
- Total error under $O(\epsilon)$ is $O\big(n(\epsilon)T(\epsilon)\big)$ – by Proposition 1

$\kappa_\star$ is the condition number

$\Delta_\star = \|I_n(z)\|^2_{H(\theta_\star)}$

Total Error = $\kappa_\star\left(1+\frac{p_\star}{\epsilon}\right)\log\left(\frac{\kappa_\star\Delta_\star}{\epsilon}\right)$

# *Result: Global Bounds*

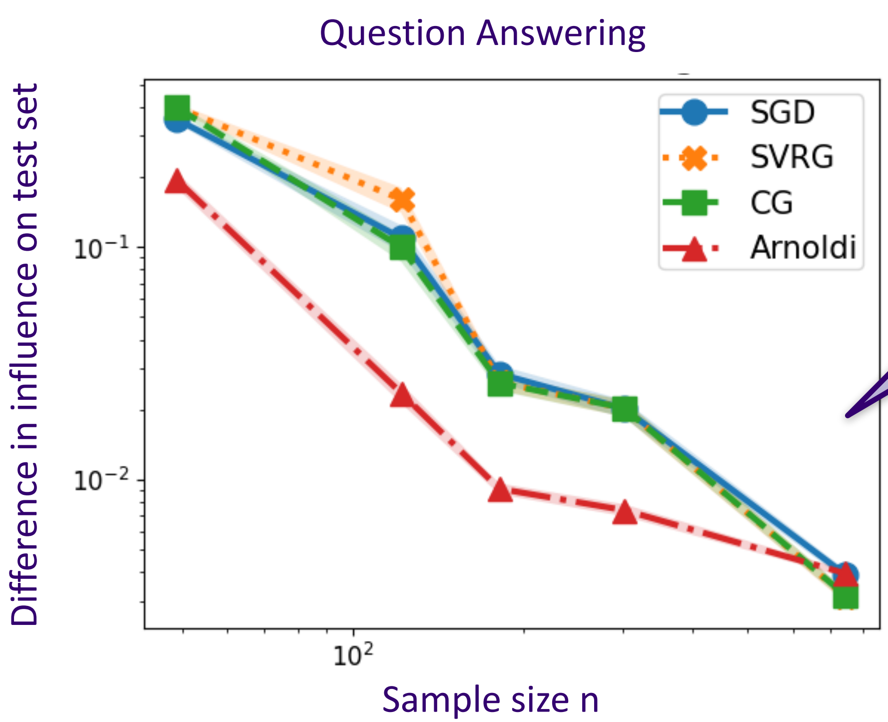
| Method | Computational Error | Total Error |
|---|---|---|
| Conjugate Gradient | $n\sqrt{\kappa_n}$ | $\frac{\kappa_\star^{3/2} p_\star^2}{\epsilon}$ |
| Stochastic Gradient Descent | $\frac{\sigma_n^2}{\epsilon} + \kappa_n$ | $\frac{\sigma_\star^2}{\epsilon} + \kappa_\star$ |
| Stochastic Variance Reduction Gradient | $(n + \kappa_n)$ | $\kappa_\star \left(1 + \frac{p_\star^2}{\epsilon}\right)$ |
| Accelerated Stochastic Variance Reduction Gradient | $(n + \sqrt{n\kappa_n})$ | $\kappa_\star \left(\sqrt{\frac{p_\star^2}{\epsilon}} + \frac{p_\star^2}{\epsilon}\right)$ |

# *Experiment: Is there always meaning?*

## Question Answering

- Input: question
- Response: factual correct answer
  - X= What country did The Laughing Cow originate?
  - Y= France
- zsRE dataset (Levy et. al., 2017)/BART-base model
- Average over 5 data points

Is this surprising?

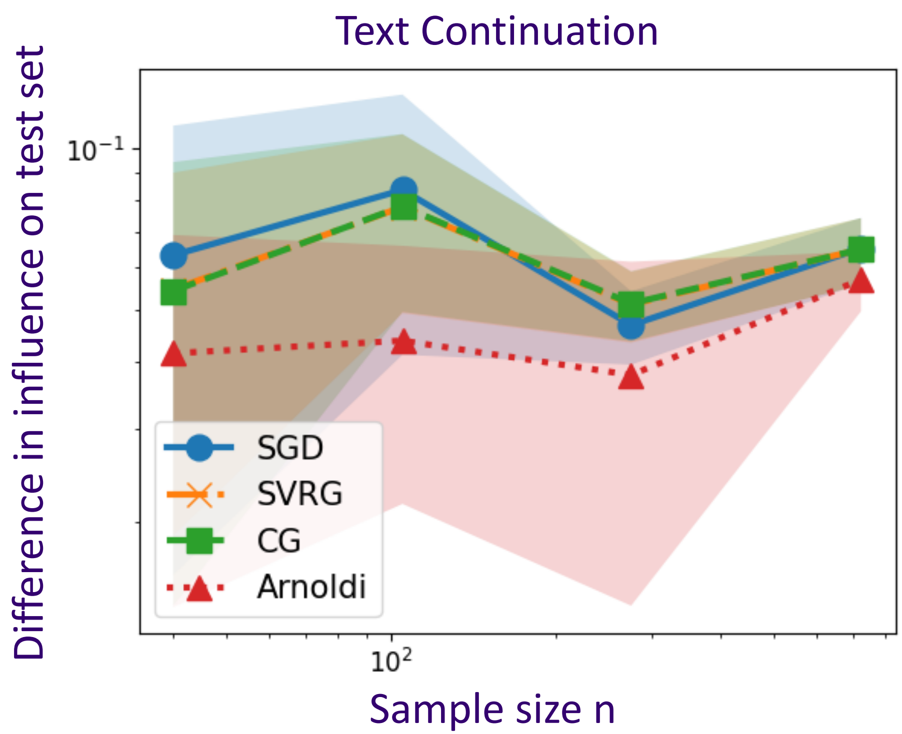



Do you think every sample would have the trend?

# *Experiment: Is there always meaning?*

## Text Continuation

- Input: Start of paragraph
- Response: 10 tokens continuation
  - X = “The interchange is considered by Popular Mechanics to be one of the…”,
  - y = “World’s 18 Strangest Roadways because of its height”
- WikiText (Merity et. al., 2017)/GPT2
- Averaged over 5 data points




Is this surprising?

Why is this graph not decreasing?

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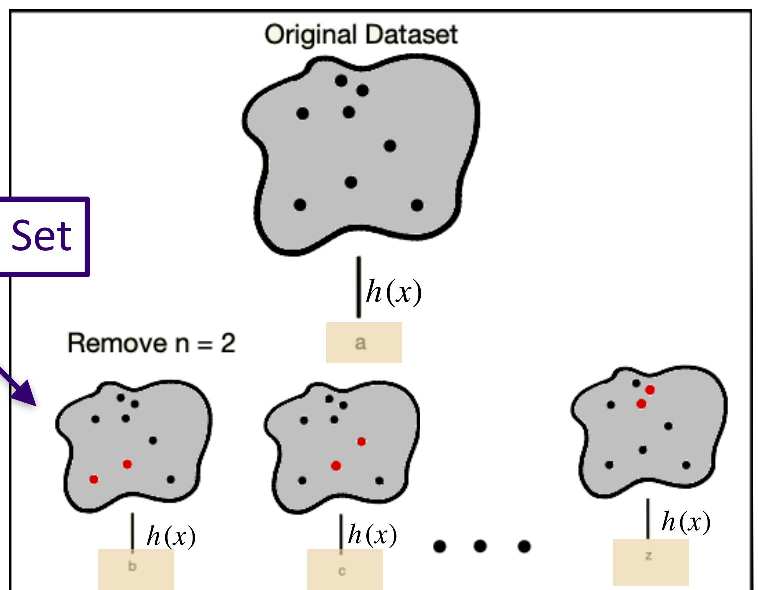
# *MIS: Definition*

## Most Influential Subset

- Given an $\alpha \in (0,1)$, and a test function $h : \mathbb{R}^p \rightarrow \mathbb{R}$

Most influential set is the subset of data (size at most $\alpha n$), which *when removed leads to largest increase* in the test function.

Maximum Influential Set




Find Largest Value
$(b - a)$ vs. $(c - a)\ldots$ vs. $(z - a)$

Maximum Influence Perturbation

# *MIS: Definition*

## Most Influential Subset

- Given n $\alpha \in (0,1)$, and a test function $h : \mathbb{R}^p \rightarrow \mathbb{R}$

Most influential subset is the subset of data (size at most $\alpha n$), which *when removed leads to largest increase* in the test function.

**Mathematically,**

$$max_{w \in W_\alpha} h(w \cdot \theta)$$

Vector of 0 and 1's → $w$

$$h\left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \right)$$

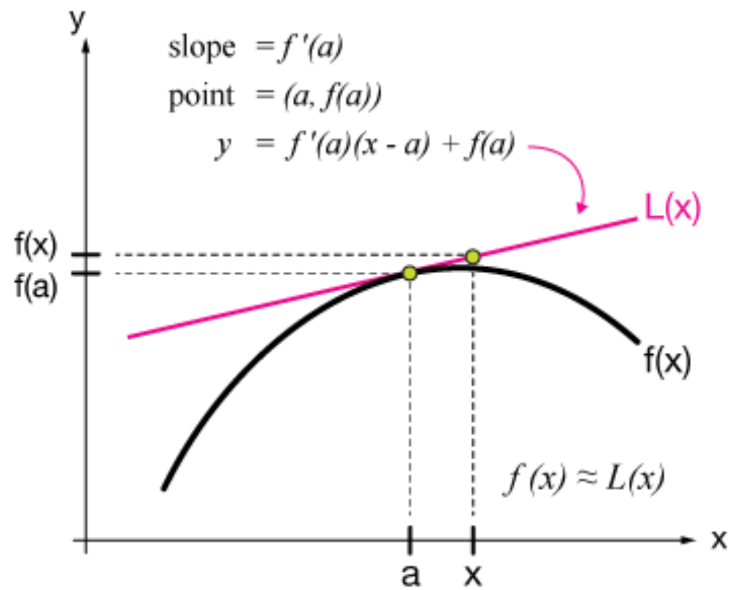
$$W_\alpha := \left\{ w \in \delta^{n-1} : \textbf{ at most } \alpha n \textbf{ elemetns of } w \textbf{ are zero and the rest are equal} \right\}$$

**Intractable as $|W_\alpha|$ grows exponentially with $n$**

N=400, m =4
1.05 * 10^9 sets to test
1 sec/run = 33 years!

# *MIS: Definition*

First-order Taylor expansion: $f(x) = f(a) + f'(a)(x - a)$




**Instead Broderick et al. (2020) use first-order Taylor expansion in $h(\theta_{n,w})$ around $w = \mathbf{1}$**

$$h(\theta_{n,w}) \approx h(\theta_n, \frac{\mathbf{1}}{n}) + \left\langle \nabla_w h(\theta_n, w) \rvert_{w=\frac{\mathbf{1}}{n}}, w - \frac{\mathbf{1}}{n} \right\rangle$$

**1** is a vector of all 1's

# *MIS: Definition*

**Instead Broderick et al. (2020) use linear *approximation***

$$h(\theta_{n,w}) \approx h(\theta_n) + \left\langle w - \frac{\mathbf{1}_n}{n}, \nabla_w h(\theta_n, w)\,|_{w=\mathbf{1}_n/n} \right\rangle$$

**Which leads to the influence of the most influential subset,**

$$I_{\alpha,n}(h) := \max_{w \in W_\alpha} \left\langle w, \nabla_w h(\theta_n, w)\,|_{w=\mathbf{1}_n/n} \right\rangle$$

**Which can be simplified using the implicit function theorem and the chain rule to a closed form**

$$I_{\alpha,n}(h) := \max_{w \in W_\alpha} \sum_{i=1}^{n} w_i v_i$$

Greedy algorithm that zeros out the largest $\alpha n$ entries of $v_i$'s!

**Where** $v_i = -\langle \nabla h(\theta_n), \underbrace{H_n(\theta_n)^{-1} \nabla \ell(Z_i, \theta_n)}_{I_n(Z_i, \theta_n)} \rangle$

# Main Results: Most Influential Subset

**Theorem 2. Suppose the added assumptions hold and the sample size $n$ satisfies the condition in Theorem 1.**

**Then with probability at least $1-\delta$**

$$\underline{\left(I_{\alpha,n}(h) - I_{\alpha}(h)\right)^2} \leq \frac{C_{M_1,M_2,M_1',M_2'}}{(1-\alpha)^2} \frac{R^2 p_\star}{\mu_\star n} \log \frac{n \vee p}{\delta}$$

- Only logarithmic dependence on $p$
- $p_\star$ is affine-invariant
- $\frac{1}{n}$ rate

# Experiment: Real Dataset

**Oregon Medicaid study (Finkestein et al., 2012)**

- Lottery from 90,000 people to sign up for Medicaid = randomization into treatment (Medicaid) and control (no Medicaid) groups
- Measured outcomes one year after treatment group received Medicaid ($\underline{n \approx 22{,}000}$)

$$y = \beta_0 + \beta_1 LOTTERY + \beta_2 X_{covariates}$$

- Test function, $h(x)$ : is $\beta_1$ significant?

Is this surprising?

| case | Original estimate | Target change | Refit estimate | Observations dropped |
|---|---|---|---|---|
| p 12m | 0.133 (0.026)* | Sign change | **-0.006 (0.025)** | 275 = 1.18% |
| | | Significance change | **0.044 (0.026)** | 162 = 0.69% |
| | | Significant sign change | -0.043 (0.024) | 381 = 1.63% |
| Health not oor 12m | 0.099 (0.018)* | Sign change | **-0.003 (0.015)** | 155 = 0.66% |
| | | Significance change | **0.027 (0.016)** | 100 = 0.43% |
| | | Significant sign change | **-0.030 (0.015)*** | 219 = 0.94% |
| | 113 (0.023)* | Sign change | **-0.006 (0.022)** | 197 = 0.84% |
| | | Significance change | **0.039 (0.022)** | 106 = 0.45% |
| | | Significant sign change | **-0.049 (0.022)*** | 291 = 1.24% |
| | 317 (0.563)* | Sign change | **-0.023 (0.535)** | 73 = 0.33% |
| | | Significance change | **1.078 (0.558)** | 10 = 0.05% |
| | | Significant sign change | -1.009 (0.521) | 144 = 0.66% |
| Not bad days physical 12m | 585 (0.606)* | Sign change | **-0.040 (0.577)** | 87 = 0.41% |
| | | Significance change | **1.131 (0.597)** | 20 = 0.09% |
| | | Significant sign change | **-1.141 (0.566)*** | 164 = 0.77% |
| Not bad days mental 12m | 2.082 (0.640)* | Sign change | **-0.062 (0.607)** | 123 = 0.57% |
| | | Significance change | **1.171 (0.625)** | 42 = 0.19% |
| | | Significant sign change | **-1.201 (0.594)*** | 212 = 0.98% |

Do you think large datasets (like the ones we use in LLM pretraining) are this sensitive to change?

On average the removal of <.05% of the data changes sign of significance!!

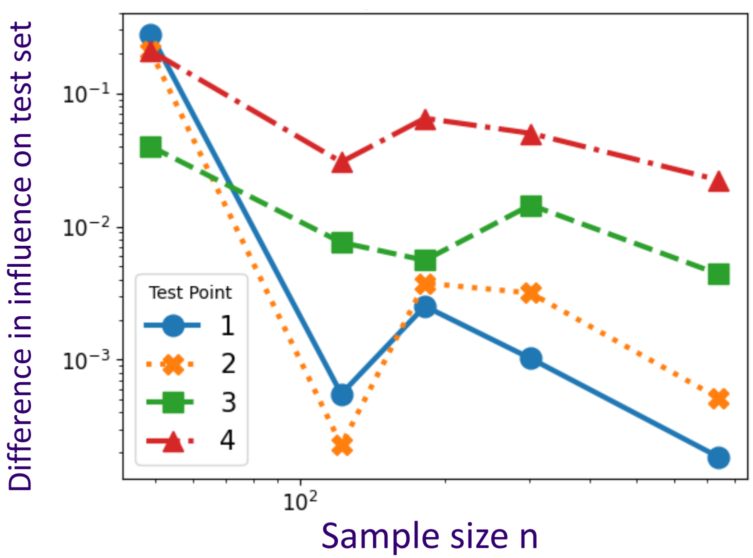
# *Experiment: Most Influential Subset*

## MIS (Question Answering)

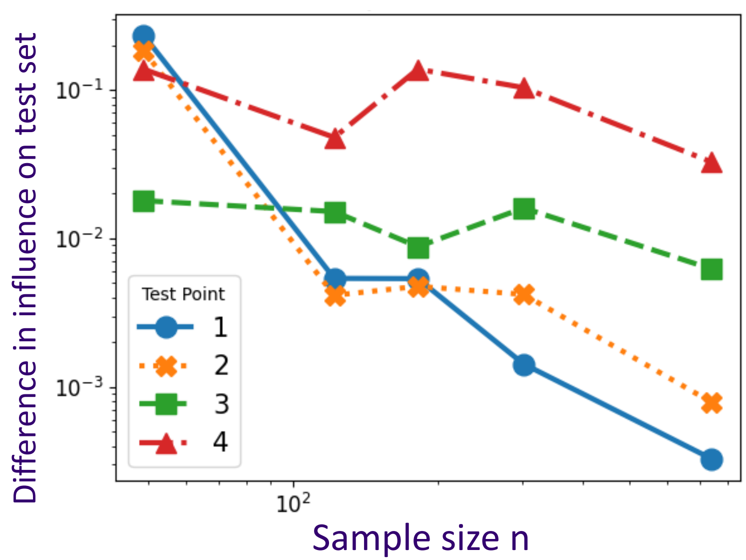
- 4 different test points (questions/answer)
- $\alpha = 0.05,.0.1$ (size of subset)
- Arnoldi method was used to approximate influence

Why is there such variety in slopes?

Alpha = 0.05

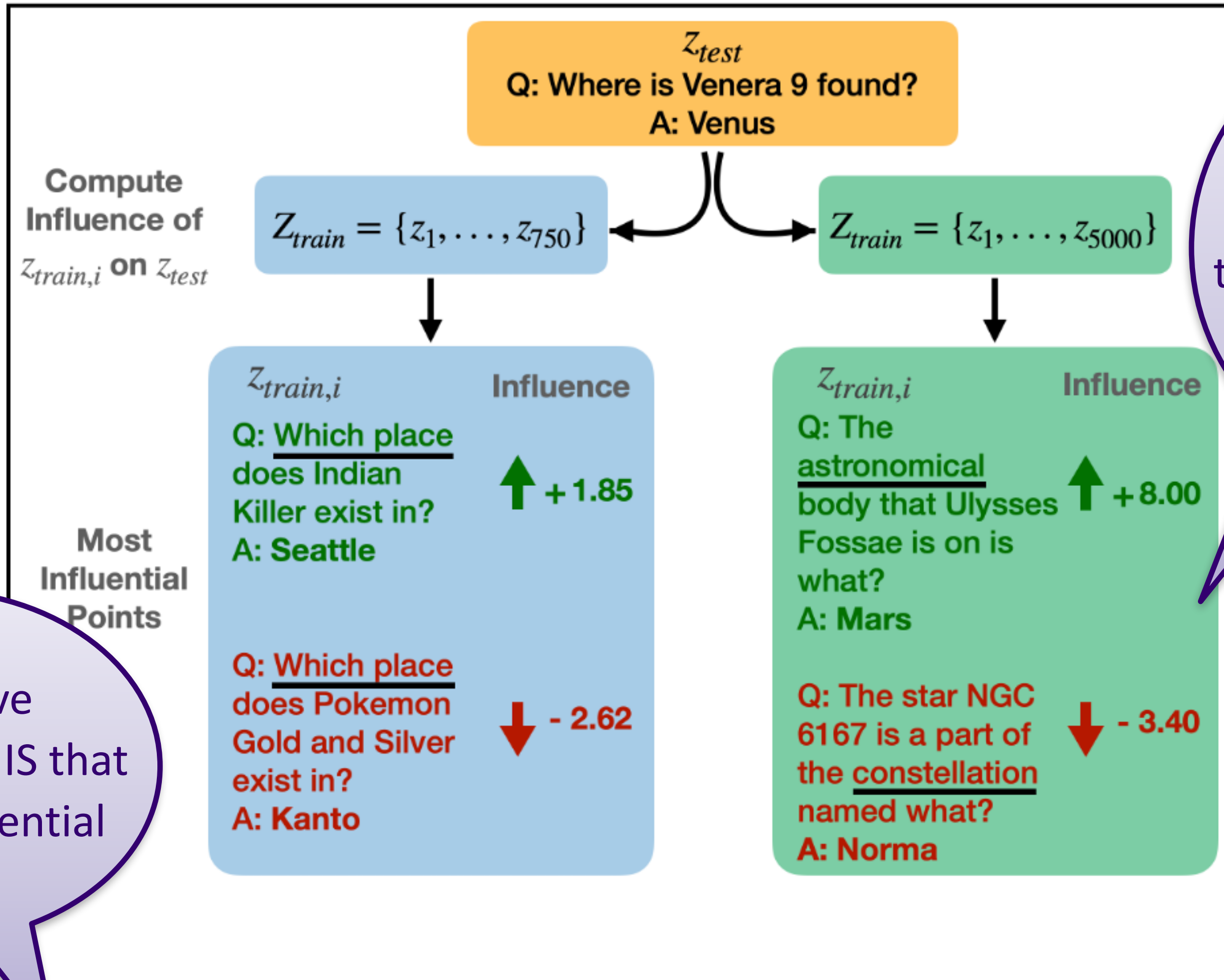



Alpha = 0.01




- Downward trend —> similar to influence of 1 datapoint

# *Experiment: Most Influential Subset*




If the MIS was perfect, how could this be used to help in NLP tasks?

Is there alternative methods to MIS that identify influential sets?

***Additional Datapoints from Smaller Train Dataset:***

The star Palomar 2 is a part of the constellation named what?

The star NGC 4349-127 is part of what constellation? …

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  - *Will there be influence in your future?*

# *Related Work in NLP*

## Influential points

- Leave one out training (data point importance)
- Saliency maps (token importance)
- Self-influence (Bejan et al. , 2023)
- Influence function for NLP…. Still in development

## Machine Unlearning

- Quark - reinforcement learning (Lu et al., 2022)
- SISA Training (Kumar et al., 2022)

**Explaining Black Box Predictions and Unveiling Data Artifacts through Influence Functions**

Xiaochuang Han, Byron C. Wallace, Yulia Tsvetkov

**INFLUENCE FUNCTIONS IN DEEP LEARNING ARE FRAGILE**

Samyadeep Basu*, Phillip Pope *& Soheil Feizi
Department of Computer Science
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AUG 27 2020

Influence Functions Do Not Seem to Predict Usefulness in NLP Transfer Learning

Author(s): Vid Kocijan and Samuel R. Bowman
Publication date: August 27 2020
Reviewer: Alex Wang
Editor: Kyunghyun Cho

# *Conclusion and Future Extensions*

Conclusion

- Presented statistical and computational guarantees for influence functions for generalized linear models
- Established the statistical consistency of most influential subsets method (Broderick et at., 2020) together with non-asymptotic bounds
- Illustrated our results on simulated and real datasets

Future Extension

- Non-convex/Non-smooth penalized M-estimation
- Application for toxicity/bias removal in NLP

# Thank You!

Full Paper

# References


R. Cook and S. Weisberg. Residuals and influence in regression.New York: Chapman and Hall, New York: Chapman Hall, 1982.

T. Broderick, R. Giordano, and R. Meager. An Automatic Finite-Sample Robustness Metric: When Can Dropping a Little Data Make a Big Difference? arXiv Preprint, 2020

D. M. Ostrovskii and F. Bach. Finite-sample analysis of M-estimators using self-concordance. Electronic Journal of Statistics, 15(1), 2021

# *Appendix Slides*

# *Algorithms: Conjugate Gradient*

**Algorithm 1** Conjugate Gradient Method to Compute the Influence Function

**Input:** vector $v$, batch Hessian vector product oracle $\mathrm{HVP}_n(u) = H_n(\theta_n)u$, number of iterations $T$

```
u_0 = 0, r_0 = -v - HVP_n(u_0), d_0 = r_0
for t = 0, ..., T - 1 do
    α_t = (d_t^⊤ r_t) / (d_t^⊤ HVP_n(d_t))
    u_{t+1} = u_t + α_t d_t
    r_{t+1} = -v - HVP_n(u_{t+1})
    β_t = (r_{t+1}^⊤ r_{t+1}) / (r_t^⊤ r_t)
    d_{t+1} = r_{t+1} + β_t d_t
return u_T
```

# *Algorithms: Stochastic Gradient Descent*

**Algorithm 2** Stochastic Gradient Descent Method to Compute the Influence Function

**Input:** vector $v$, Hessian vector product oracle $\text{HVP}(i, u) = \nabla^2 \ell(z_i, \theta_n)u$, number of iterations $T$, learning rate $\gamma$

$u_0 = 0$
**for** $t = 0, ..., T - 1$ **do**
    Sample $i_t \sim \text{Unif}([n])$
    $u_{t+1} = u_t - \gamma(\text{HVP}(i_t, u_t) + v)$
**return** $u_T$

# Algorithms: Stochastic Variance Reduction Gradient

**Algorithm 4** Stochastic Variance Reduced Gradient Method to Compute the Influence Function

**Input:** vector $v$, Hessian vector product oracle $\text{HVP}(i, u) = \nabla^2 \ell(z_i, \theta_n) u$, number of epochs $S$, number of iterations per epoch $T$, learning rate $\gamma$

$u_T^{(0)} = 0$
**for** $s = 1, 2, ..., S$ **do**
  $u_0^{(s)} = u_T^{(s-1)}$
  $\tilde{u}_0^{(s)} = \frac{1}{n} \sum_{i=1}^{n} \text{HVP}(u_0^{(s)}) - v$
  **for** $t = 0, ..., T-1$ **do**
    Sample $i_t \sim \text{Unif}([n])$
    $u_{t+1}^{(s)} = u_t^{(s)} - \gamma(\text{HVP}(i_t, u_t^{(s)}) - \text{HVP}(i_t, u_0^{(s)}) + \tilde{u}_0^{(s)})$
**return** $u_T^{(S)}$

# Algorithms: Arnoldi

**Algorithm 5** Arnoldi Method to Compute the Influence Function (Schioppa et al., 2022)

**Input:** vector $v$, test function $h$, initial guess $u_0$, batch Hessian vector product oracle $\mathrm{HVP}_n(u) = H_n(\theta_n)u$, number of top eigenvalues $k$, number of iterations $T$

**Output:** An estimate of $\langle \nabla h(\theta), H_n(\theta_n)^{-1} v \rangle$

Obtain $\Lambda, G = \textsc{Arnoldi}(u_0, T, k)$ ▷ Cache the results for future calls<br>
**return** $\langle G\nabla h(\theta), \Lambda^{-1} G v \rangle$

**procedure** $\textsc{Arnoldi}(u_0, T, k)$<br>
    $w_0 = 1 = u_0 / \|u_0\|_2$<br>
    $A = \mathbf{0}_{T+1 \times T}$<br>
    **for** $t = 1, ..., T$ **do**<br>
        Set $u_t = \mathrm{HVP}_n(w_t) - \sum_{j=1}^{t} \langle u_t, w_j \rangle \, w_j$<br>
        Set $A_{j,t} = \langle u_t, w_j \rangle$ for $j = 1, \ldots, t$ and $A_{t+1,t} = \|u_t\|_2$<br>
        Update $w_{t+1} = u_t / \|u_t\|$<br>
    Set $\tilde{A} = A[1:T, \ :] \in \mathbb{R}^{T \times T}$ (discard the last row)<br>
    Compute an eigenvalue decomposition $\tilde{A} = \sum_{j=1}^{T} \lambda_j e_j e_j^\top$ with $\lambda_j$'s in descending order<br>
    Define $G : \mathbb{R}^p \to \mathbb{R}^k$ as the operator $Gu = \left(\langle u, W^\top e_1 \rangle, \cdots, \langle u, W^\top e_k \rangle\right)$, where $W = (w_1^\top; \cdots; w_T^\top) \in \mathbb{R}^{T \times p}$<br>
    **return** diagonal matrix $\Lambda = \mathrm{Diag}(\lambda_1, \cdots, \lambda_k)$ and the operator $G$

# Computational Results: CG

**Proposition 1. Consider the setting of Theorem 1, and let $\mathcal{G}$ denote the event under which its conclusions hold. Let $\hat{I}_n(\theta)$ be an estimate of $I_n(\theta)$ that satisfies $\mathbb{E}\left[\left\|\hat{I}_n(z) - I_n(z)\right\|^2_{H_n(\theta_n)} \middle| Z_{1:n}\right] \leq \epsilon$.**

**Then**

$$\mathbb{E}\left[\left\|\hat{I}_n(z) - I_n(z)\right\|^2_{H_\star} \middle| \mathcal{G}\right] \leq 8\epsilon + C\frac{R^2 p_\star^2}{\mu_\star n}\log^3\left(\frac{p}{\delta}\right)$$

Example: Conjugate Gradient

- Requires $T_n(\epsilon) := \sqrt{k_n}\log(\left\|I_n(z)\right\|^2_{H_n(\theta_n)}/\epsilon)$ iterations to return an $\epsilon$-approximate minimizer.
- Each iteration requires $n$ Hessian-vector products
- To make statistical error to be smaller than $\epsilon$, $n \geq n(\epsilon) = \tilde{O}\left(\frac{R^2 p_\star^2}{\mu_\star \epsilon}\right)$
- Total error under $O(\epsilon)$ is $O\left(n(\epsilon)T(\epsilon)\right)$ – by Proposition 1

# *Experiment: Most Influential Subset*

## MIS Test Questions

1. What position did Víctor Vázquez Solsona play? ***- midfielder***
2. The nationality of Jean-Louis Laya was what? ***- French***
3. Where is Venera 9 found? ***- Venus***
4. Who set the standards for ISO 3166-1 alpha-2? ***- International Organization for Standardization***
5. In which language Nintendo La Rivista Ufficiale monthly football magazine reporting? ***- Italian***